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Simultaneous Best Approximations with Two Polynomials

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1. THE PROBLEM

The following problem was posed by T. J. Rivlin [*1*]: Let $\mathscr{C}$ stand for the space of continuous real valued functions on the interval [0,1]; let $E_j(f)$ represent the Chebyshev degree of approximation to f by algebraic polynomials of degree j. Characterize the n-tuples $\{p_0, p_1, \ldots, p_{n-1}\}$ of algebraic polynomials p_j, where $\deg p_j = j$, which have the property that there is a function $f \in \mathscr{C}$ such that

$$E_j(f) = \|f - p_j\| \tag{1}$$

for $j = 0, 1, 2, \ldots, n-1$, the norm on the right side being the uniform norm.

In this note we prove the following related

THEOREM. *Given polynomials p_m and p_n, $0 \leqslant m < n$, there is a function $f \in \mathscr{C}$ which satisfies* (1) *for the integers $j = m$ and $j = n$ if and only if the polynomial $p_n - p_m$ changes sign at least $m + 1$ times in* [0, 1].

The linear case $m = 0$, $n = 1$ was proved by Deutsch, Morris and Singer in [*2*]. Our only excuse for reproving this case is that our proof is very short. The necessity of the condition in the theorem was stated and proved by Rivlin [*1*].

2. PROOF OF THE CASE $m = 0$, $n = 1$

To verify the sufficiency of the condition let us consider polynomials $p_0(x) = c$ and $p_1(x) = ax + b$ ($a \neq 0$) such that $at + b = c$ for some t, $0 < t < 1$. Then the equality

$$\tfrac{1}{2}[p_1(t - \delta) + p_1(t + \delta)] = c$$

holds for each number δ, and we fix a δ such that $0 < \delta < \min\{t, 1 - t\}$. Let us suppose that $a > 0$ and put

$$\alpha = \max\{|a + b - c|, |b - c|\}.$$

If $f \in \mathscr{C}$ is the piecewise linear function with vertices

$$(0, c-\alpha) \quad (t-\delta, c-\alpha) \quad (t+\delta, c+\alpha) \quad (1, p_1(t+\delta)),$$

(see Fig. 1), then (1) is easily seen to hold for $j = 0, 1$. The desired f in case $a < 0$ is obtained from the above construction by the substitution $x = 1 - y$.

The necessity of the asserted sign change is demonstrated as follows: let a function $f \in \mathscr{C}$ and polynomials p_0, p_1 satisfy (1). If we put

$$g_j = f - p_j \qquad (j = 0, 1),$$

then it is an elementary observation that

$$\max g_j + \min g_j = 0 \qquad (j = 0, 1), \tag{2}$$

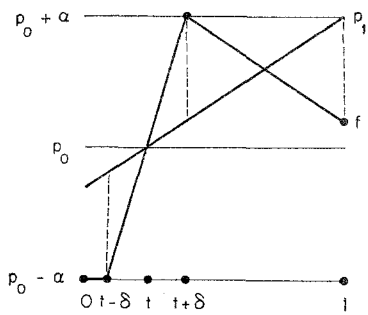

FIG. 1.

where the maximum and minimum are taken over $[0, 1]$. If, say, $g_1 > g_2$ on $(0, 1)$, then $\max g_1 > \max g_2$ and $\min g_1 > \min g_2$, so that

$$\max g_1 + \min g_1 > \max g_2 + \min g_2.$$

But this implies that (2) fails to be true for g_1 or g_2. Clearly, one arrives at the same conclusion if $g_2 > g_1$ on $(0, 1)$ and the necessity of the sign change is hereby proved.

3. Proof of the Case $m = 0$, $n > 1$

Let $p_n(x_0) = c$ for some point $0 < x_0 < 1$. Then there are points $0 < x_1 < x_0 < x_2 < 1$ such that

$$\tfrac{1}{2}[p_n(x_1) + p_n(x_2)] = c;$$

we may assume without loss of generality that $p_n(x_2) > c$. We introduce the polynomials

$$\begin{aligned} p_n^-(x) &= p_n(x) - [p_n(x_1) + \|p_n\| - c], \\ p_n^+(x) &= p_n(x) + [p_n(x_1) + \|p_n\| - c], \end{aligned} \tag{3}$$

satisfying

$$\tfrac{1}{2}[p_n^-(x) + p_n^+(x)] = p_n(x).$$

Let

$$k = \begin{cases} n+3 & \text{if } n \text{ is odd,} \\ n+4 & \text{if } n \text{ is even,} \end{cases}$$

and fix points

$$x_1 = u_1 < u_2 < u_2 < \ldots < u_{k-1} < u_k = x_2$$

(see Fig. 2). The function

$$g(x) = \begin{cases} \dfrac{u_{2j} - x}{u_{2j} - u_{2j-1}} p_n^-(x) + \dfrac{x - u_{2j-1}}{u_{2j} - u_{2j-1}} p_n^+(x) \\ \qquad\qquad \left(u_{2j-1} \leqslant x \leqslant u_{2j}, j = 1, 2, \ldots, \dfrac{k}{2}\right) \\ \dfrac{u_{2j+1} - x}{u_{2j+1} - u_{2j}} p_n^+(x) + \dfrac{x - u_{2j}}{u_{2j+1} - u_{2j}} p_n^-(x) \\ \qquad\qquad \left(u_{2j} \leqslant x \leqslant u_{2j+1}, j = 1, 2, \ldots, \dfrac{k}{2} - 1\right) \end{cases} \tag{4}$$

which is defined and continuous on the interval $[x_1, x_2]$, is best approximated in the class of all n-degree algebraic polynomials by $p_n(x)$. This follows from the facts that

$$p_n(u_{2j-1}) - g(u_{2j-1}) = p_n(u_{2j-1}) - p_n^-(u_{2j-1}) = p_n(x_1) + \|p_n\| - c \qquad \left(j = 1, 2, \ldots, \frac{k}{2}\right),$$

$$p_n(u_{2j}) - g(u_{2j}) = p_n(u_{2j}) - p_n^+(u_{2j}) = -[p_n(x_1) + \|p_n\| - c] \qquad \left(j = 1, 2, \ldots, \frac{k}{2} - 1\right),$$

and

$$\|g - p_n\| = p_n(x_1) + \|p_n\| - c,$$

as shown by a simple calculation. We let

$$f(x) = \begin{cases} \dfrac{x_1 - x}{x_1} p_n(x) + \dfrac{x}{x_1} p_n^-(x) & (0 \leqslant x \leqslant x_1) \\ g(x) & (x_1 \leqslant x \leqslant x_2) \\ \dfrac{1 - x}{1 - x_2} p_n^+(x) + \dfrac{x - x_2}{1 - x_2} p_n(x) & (x_2 \leqslant x \leqslant 1) \end{cases}$$

Then

$$\|f-p_n\| = \|g-p_n\|$$

so that $E_n(f) = \|f-p_n\|$. To see that $E_0(t) = \|f-c\|$ we observe from (3) that

$$f(x_1) = p_n^-(x_1) = c - \|p_n\|,$$
$$f(x_2) = p_n^+(x_2) = c + \|p_n\|,$$

and a routine calculation shows that

$$-\|p_n\| \leqslant f(x) - c \leqslant \|p_n\| \qquad (0 \leqslant x \leqslant 1).$$

Necessity is proved just as in the previous case.

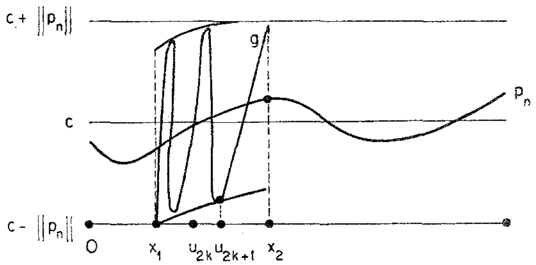

FIG. 2.

4. PROOF OF THE CASE $m > 0, n > 1$

Consider arbitrary polynomials p_m and p_n, with $1 \leqslant m < n$, such that $p_n - p_m$ changes sign at least $m+1$ times in $[0,1]$. Then there are points

$$0 < t_1 < t_2 < \ldots < t_{m+1} < 1$$

such that $p_m(t_j) = p_n(t_j)$ for $j = 1, 2, \ldots, m+1$. Letting $t_0 = 0$ and $t_{m+2} = 1$, we put

$$\alpha_j = \max_{t_j \leqslant x \leqslant t_{j+1}} |p_m(x) - p_n(x)|,$$

$$\alpha = \tfrac{1}{2} \min\{\alpha_0, \alpha_1, \ldots, \alpha_{m+1}\},$$

$$\beta = \max\{\|p_m\|, \|p_n\|\}.$$

Suppose $p_n(x) \geqslant p_m(x)$ for $t_1 \leqslant x \leqslant t_2$ and consider the polynomials

$$p_n^-(x) = p_n(x) - \beta,$$
$$p_n^+(x) = p_n(x) + \beta,$$
$$p_m^-(x) = p_m(x) - \alpha - \beta,$$
$$p_m^+(x) = p_m(x) + \alpha + \beta.$$

Then each of the polynomials $p_n^- - p_m^-$ and $p_n^+ - p_m^+$ also changes sign at least $m+1$ times in [0, 1]. Thus, there are $m+2$ points x_j,

$$x_j < t_j < x_{j+1}, \qquad j = 1, 2, \ldots, m+1,$$

such that

$$p_m^-(x_{2k-1}) = p_n^-(x_{2k-1}) \tag{5}$$

and

$$p_m^+(x_{2k}) = p_n^+(x_{2k}) \tag{6}$$

for all admitted values of k. The point x_2 can be so chosen that

$$p_m^-(x) \leqslant p_n^-(x) < p_n^+(x) \leqslant p^+(x), \qquad (x_1 \leqslant x \leqslant x_2).$$

On the interval $[x_1, x_2]$ the situation is now similar to that in Case 3. Thus, let g be defined as in (4). Then p_n is its best approximation in the class of algebraic polynomials of degree n.

Put for each $x \in [0, 1]$,

$$q^-(x) = \max\{p_m^-(x), p_n^-(x)\},$$
$$q^+(x) = \min\{p_m^+(x), p_n^+(x)\}.$$

Then

$$q^-(x_{2k-1}) = p_m^-(x_{2k-1}) = p_n^-(x_{2k-1}),$$
$$q^+(x_{2k}) = p_m^+(x_{2k}) = p_n^+(x_{2k})$$

(see (5) and (6)). In a manner similar to that in Case 3 we construct a function $f \in \mathscr{C}$ such that

$$q^-(x) \leqslant f(x) \leqslant q^+(x), \tag{7}$$

$$f(x) = g(x) \qquad \text{for } x_1 \leqslant x \leqslant x_2, \tag{8}$$

$$\left.\begin{aligned} f(x_{2k-1}) &= q^-(x_{2k-1}), \\ f(x_{2k}) &= q^+(x_{2k}). \end{aligned}\right. \tag{9}$$

The conditions (7) and (8) guarantee that p_n is also the best approximation of f by algebraic polynomials of degree n. Conditions (7) and (9) show that the best approximation to f from among the algebraic polynomials of degree m is p_m: the calculations justifying this conclusion are routine and therefore omitted here.

For the necessity of the asserted sign changes we refer this time to Rivlin's proof in [*1*].

References

1. T. J. Rivlin in Proc. Coll. on Abstract Spaces and Approximation (Oberwolfach, July 1968), Birkhäuser Verlag (to appear.)

2. F. Deutsch, P. D. Morris and I. Singer, On a problem of T. J. Rivlin in approximation theory. *J. Approx. Theory* (to appear).